A Quick Derivation relating altitude to air pressure

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Using a barometer to measure altitude is a well established technique. The idealized theory for doing this is easily expressed. This derivation aims to make the concepts involved easily and rapidly available to a technical audience.

Introduction

This derivation is based on a subset of the *International Standard Atmosphere* (ISA) model formulated by the *International Civil Aviation Organization* (ICAO). The main assumptions are hydrostatic equilibrium, perfect gas, gravity independent of altitude, and constant lapse rate. Zero altitude is measured from mean sea level, which is defined in terms of the gravitational potential energy, and therefore varies relative to the geodetic ellipsoid.

The ICAO atmospheric model has been updated from time to time and now exists in several versions including the *International Standards Organization* 1973 and the *US Standard Atmosphere* 1976. The most recent revision of the ISA at the time of writing is due to the ICAO 1993. For the lower atmosphere, the differences between all these models appears to be inconsequential, though this has not been verified.

Here is a full listing of the atmospheric parameters used in this document (SI units):

Symbol	Value	Unit	Description
P_0	101325	Pa	pressure at zero altitude (base pressure)
T_0	288.15	K	temperature at zero altitude
g	9.80665	m/s^2	acceleration due to gravity
L	-6.5×10^{-3}	K/m	lapse rate
R	287.053	J/(kg K)	gas constant for air
Rh	0 %	dimensionless	relative humidity

The lapse rate is defined as the rate of temperature increase in the atmosphere with increasing altitude. For the ISA the lapse rate near the ground is assumed to be -6.5 °C / 1000 m. The negative sign indicates a decrease of temperature with altitude.

The ISA assumes a constant lapse rate between 0 and 11 km altitude. The upper bound marks the beginning of the stratosphere, where the atmospheric temperature becomes relatively constant with respect to altitude.

Limitations

The ISA, or this subset of it, fails to accurately model the real atmosphere is several ways.

In this model the lapse rate is assumed to be single constant. This is fairly accurate up to about 11km. Above 11km the equations derived here will probably fail to accurately predict the pressure-altitude relationship. The situation can be improved by dividing the atmosphere piece−wise into layers and using a different lapse rate for each layer. This is done in the full ISA model, but it is not attempted here.

In the real atmosphere significant variation is observed in base pressure, temperature, and lapse rate. Lapse rate can be non-constant, for example when there is a temperature inversion near the ground, or the altitude of the stratospheric temperature boundary can vary significantly.

The assumption of constant gravity, which ignores the variation of gravitational force with altitude, is generally consistent with the accuracy of the model and causes a very minor decrease in accuracy.

The assumption of hydrostatic equilibrium is usually a good one providing that averages are used to mask fluctuations due to short term winds. Certain storms can produce prolonged disequilibrium generally accompanied by severe weather conditions.

The perfect gas assumption is highly accurate for air, however the behavior of a perfect gas is influenced by the constant R, which in turn depends on mean molecular weight. The composition of the lower atmosphere is approximately constant, but a very wet atmosphere may have a water vapor content high enough to appreciably lower the atmospheric density, thus changing the R value and influencing the accuracy of these calculations.

The parameters chosen for the ISA are based on averages near 45° latitude. The accuracy will decline near the poles or the equator.

To match the present model to real atmospheric conditions several refinements are possible. The easiest is to substitute the real values of base pressure and temperature for those assumed by the standard atmosphere. This information is provided by most airports and weather stations. A second step is to use the actual lapse rate rather than the assumed one. Unfortunately lapse rate is hard to measure and for now rarely available.

The most general solution, which encompasses most of the above considerations, is to fit the model to independent pressure and altitude measurements or other data, as soon as they are available, thus constantly calibrating and refining the model. This approach deserves consideration when high accuracy is desired.

■ Derivation

In hydrostatic equilibrium, the change in pressure over an infinitesimal change in altitude must oppose the gravitational force on the air in that infinitesimal layer, that is

$$\frac{dP}{dz} == -\rho\, g \tag{1}$$

where P is pressure, z altitude, ρ air density, and g is the gravitational acceleration

The ideal gas law states

$$P == \rho\, R\, T \tag{2}$$

where R is the gas constant for air and T the temperature

Combining these equations, ρ can be eliminated

$$\frac{dP}{dz} == -\frac{g}{R\,T}\, P \tag{3}$$

For constant T and g, this is a first order linear differential equation, the solution is

$$z == -\frac{R\,T}{g}\,\text{Log}\left[\frac{P}{P_0}\right] \tag{4}$$

This equation is known as the hypsometric equation. It relates the pressure ratio to altitude under the assumptions of constant temperature and gravity.

The hypsometric equation is *not very satisfactory* because it assumes zero lapse rate. It is mentioned here because it is often cited in the literature. Don't use it for altitude determination unless the approximation of constant temperature is an acceptable one.

To get a better relation, the temperature must be taken as a variable. In general this leads to a non-linear differential equation, but in the specific case of a linear change of temperature with altitude, that is a constant lapse rate, the equation is tractable. Starting with

$$\frac{dP}{dz}[z] == -\frac{1}{R}\,\frac{g[z]}{T[z]}\,P[z] \tag{5}$$

The solution can be found by integration, it is

$$P[z] == P_0\,\text{Exp}\left[-\frac{1}{R}\left(\int_0^z \frac{g[\zeta]}{T[\zeta]}\,d\zeta\right)\right] \tag{6}$$

where ζ is just a dummy variable for the integration

Assuming g is constant with respect to altitude and introducing the linearized temperature profile via the lapse rate, the integral becomes

$$g\int_0^z \frac{d\zeta}{T_0 + L\,\zeta} == \frac{g}{L}\,(\text{Log}[T_0 + L\,z] - \text{Log}[T_0]) \tag{7}$$

where L is the lapse rate

Substituting this into the expression for P[z] (6) and solving for z gets

$$z == \frac{T_0}{L}\left(\left(\frac{P}{P_0}\right)^{-L\,R/g} - 1\right) \tag{8}$$

This is our final, simplified, expression for altitude in terms of atmospheric pressure. Remember when using this expression that L near the ground is a negative number.

■ Example

The 1996 edition of the CRC handbook gives this expression for pressure in terms of altitude in the standard atmosphere

$$P == 100 * \left(\frac{44331.514 - z}{11880.516} \right)^{1/0.1902632} \quad (9)$$

where again P is pressure in Pascals and z is altitude in meters

This expression can be solved for z getting

$$z == 44331.5 - 4946.62\, P^{0.190263} \text{ (CRC)} \quad (10)$$

Using equation (8) derived above, and substituting the standard atmosphere parameters gives

$$z == 44330.8 - 4946.54\, P^{0.1902632} \text{ (Equation 8)} \quad (11)$$

Equations (10) and (11) are quite similar. Plotting them together shows this more clearly

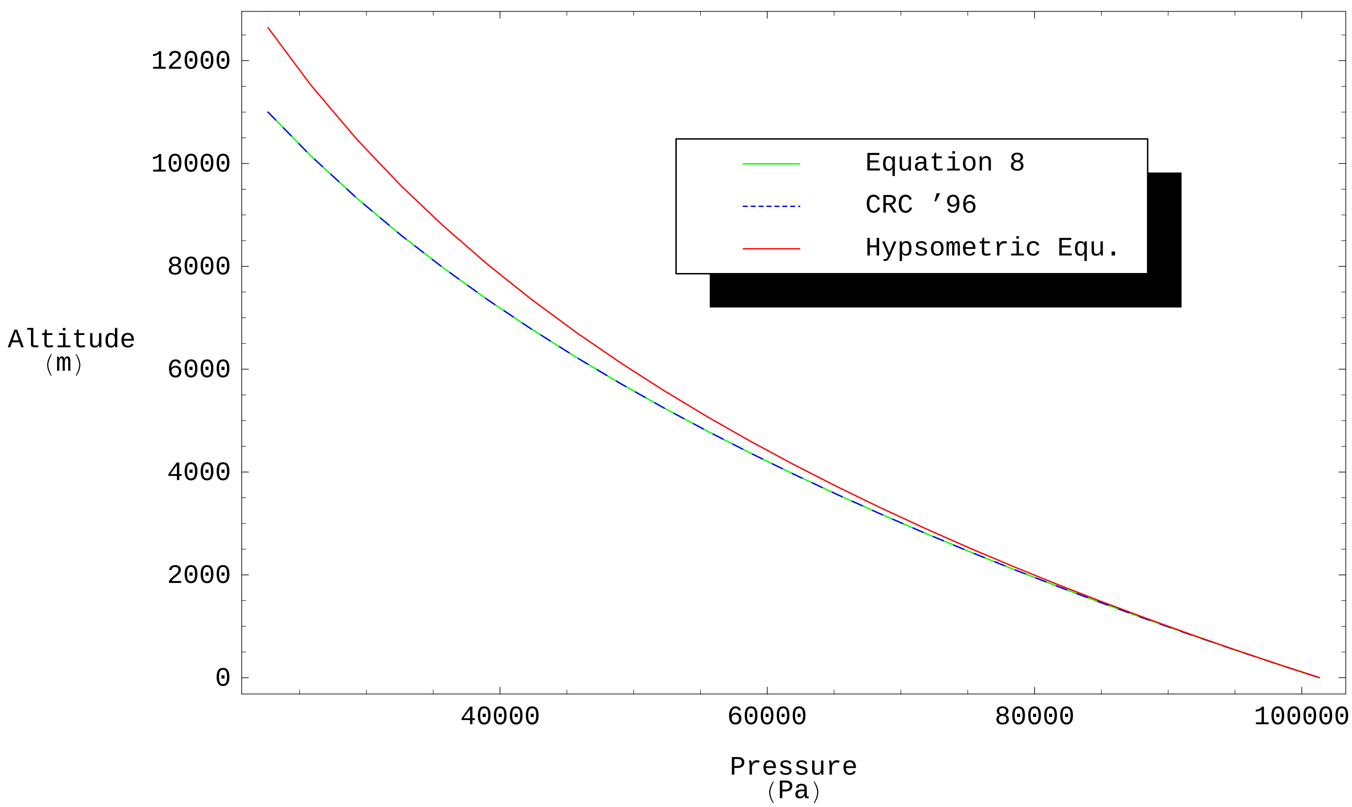

Figure 1

Graphically the agreement between our derivation in equation (11) and the CRC formula (10) is almost perfect.

The hypsometric equation (4) has been included for comparison. The deviation is obvious, particularly at higher altitudes, where the temperature difference is greater.